International Journal of Theoretical Physics, Vol. 32, No. 6, 1993

Equivalence of Dirac and Maxwell Equations and Quantum Mechanics

Jayme Vaz, Jr.[1,2] **and Waldyr A. Rodrigues, Jr.**[1]

Received March 5, 1992

In this paper we present an analysis of the possible equivalence of Dirac and Maxwell equations using the Clifford bundle formalism and compare it with Campolattaro's approach, which uses the traditional tensor calculus and the standard Dirac covariant spinor field. We show that Campolattaro's intricate calculations can be proved in few lines in our formalism. We briefly discuss the implications of our findings for the interpretation of quantum mechanics.

1. INTRODUCTION

The Maxwell equations and the Dirac equation are among the most celebrated equations of physics. Several presentations of the Maxwell equations in (matrix) Dirac-like "spinor" form can be found in the literature [see Rodrigues and de Oliveira (1990) for discussion], some of them motivated to give a "first quantization" interpretation of Maxwell fields. The possibility of an intimate relationship between the Maxwell and Dirac fields is an object of serious speculation since it could provide an answer to a long-standing question: what is an electron?

That possibility has been considered by us in two previous papers (Vaz and Rodrigues, 1992; Rodrigues and Vaz, 1992) by using the Clifford bundle formalism. Among the many advantages of this approach we can consider its simplicity and the fact that with it the Maxwell and Dirac fields are represented by objects of the same mathematical nature. In Vaz and Rodrigues (1992) we show that starting from the *free* Maxwell equations

[1]Departamento de Matemática Aplicada, IMECC-UNICAMP, Universidade Estadual de Campinas, 13083 Campinas, SP, Brazil.
[2]Departamento de Ciências Naturais, Fundação de Ensino Superior de São João del Rei, 36300 São João del Rei, MG, Brazil.

0020-7748/93/0600-0945$07.00/0

and by writing[3]

$$F = b\psi\gamma^1\gamma^2\psi^* \tag{1}$$

for the electromagnetic field [where ψ is a Dirac–Hestenes (DH) spinor field] that the nonlinear DH equation[4]

$$\gamma^\mu\partial_\mu\psi\gamma^1\gamma^2 + \mathscr{F}(\psi) = 0 \tag{2}$$

with

$$\mathscr{F}(\psi) = \gamma^\mu\psi\gamma^1\gamma^2(\partial_\mu\psi^*)\psi(\psi^*\psi)^{-1} \tag{3}$$

is equivalent to the free Maxwell equations, and that there are two solutions ψ of equation (2) which satisfy the DH equation (which is the representative of the standard Dirac equation in the Clifford bundle over Minkowski spacetime). These two solutions were naturally identified as "electron" and "positron" solutions. In Rodrigues and Vaz (1992) we generalized our approach in order to obtain *localized* "electron" solutions. In Section 2 we briefly review our approach.

The possibility of an intimate correspondence between Maxwell and Dirac fields has also been considered by Campolattaro (1980*a*,*b*, 1990). Campolattaro (1980*a*) deduced a nonlinear Dirac-like spinor equation (for the usual standard Dirac covariant spinor field) equivalent to the Maxwell equations. For the case where the electromagnetic current $J = 0$ the equation is (see footnote 4)

$$\gamma^\mu\partial_\mu\Psi = -i\gamma^\mu\frac{e^{\gamma^5\alpha}}{\rho}\{\mathrm{Im}(\partial_\mu\bar{\Psi}\Psi) - \gamma^5\,\mathrm{Im}(\partial_\mu\bar{\Psi}\gamma^5\Psi)\}\Psi \tag{4}$$

where

$$\mathrm{Im}(\partial_\mu\bar{\Psi}\gamma^5\Psi) = -\tfrac{1}{2}(\partial_\mu\bar{\Psi}i\gamma^5\Psi - \bar{\Psi}i\gamma^5\partial_\mu\Psi) \tag{5}$$

$$\mathrm{Im}(\partial_\mu\bar{\Psi}\Psi) = \frac{1}{2i}(\partial_\mu\bar{\Psi}\Psi - \bar{\Psi}\partial_\mu\Psi) \tag{6}$$

and α is the "complexion" (Misner and Wheeler, 1957) of the field $F^{\mu\nu}$ in the given point. We also have

$$\cos\alpha = \frac{\bar{\Psi}\Psi}{\rho}, \qquad \sin\alpha = \frac{\bar{\Psi}\gamma^5\Psi}{\rho} \tag{7}$$

$$\rho^2 = (\bar{\Psi}\Psi)^2 + (\bar{\Psi}\gamma^5\Psi)^2 \tag{8}$$

so that $\Omega_1 = \bar{\Psi}\Psi$ and $\Omega_2 = \bar{\Psi}\gamma^5\Psi$ are the invariants in the Dirac theory.

[3]This is always possible according to the theorem of Rainich–Misner–Wheeler (see Appendix B).

[4]In equations (1)–(3), etc., γ^μ are the generators of the local Clifford algebra $\mathbb{R}_{1,3}$ of the Clifford bundle. The γ^μ in equation (4), etc., are the usual Dirac matrices in the standard representation. We shall use the same symbol for both objects, since no confusion appears in the text.

Now if we compare equations (2) and (4) we see that they look very different. However, Campolattaro started his demonstration of the equivalence between equation (4) and the Maxwell equations by proving that any given electromagnetic field $F^{\mu\nu}$ can be written as

$$F^{\mu\nu} = \tfrac{1}{2}\bar{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi \tag{9}$$

where $\gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}[\gamma^{\mu}, \gamma^{\nu}] = \frac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})$.

Despite the apparent difference between equations (2) and (4) and the apparent similarity between equations (1) and (9), a difference exists. In fact, while ψ in equations (1)-(2) is a DH spinor field, i.e., an operator spinor field according to Figueiredo *et al.* (1990*a*,*b*), Ψ in equations (4) and (9) is a standard Dirac covariant (SDC) spinor field, according to Figueiredo *et al.* (1990*a*, *b*). Thus $\bar{\Psi}$ is the usual Dirac conjugate $\bar{\Psi} = \psi^{+}\gamma^{0}$. In Rodrigues and Oliveira (1990) it is shown that Ψ can be identified with $\Psi = \psi e$, where $e = \frac{1}{2}(1+\gamma^{0}) \in \sec \mathscr{C}\ell(M, \hat{g})$ is a global idempotent field.

It is reasonable to suppose that equations (2) and (4) are different representations for the same equation. This is indeed the case. We prove this explicitly in Section 3, and since these two equations are the same, (4) has *plane wave* solutions, as we shall see. It is very difficult to see this fact directly from equation (4), but from equation (2) this is a trivial task. Also, with that proof, our theory (Vaz and Rodrigues, 1992; Rodrigues and Vaz, 1992) can now be translated into "traditional" mathematical terms.

2. THE CLIFFORD BUNDLE APPROACH

Let $\mathscr{C}\ell(M, \hat{g})$ be the Clifford bundle of differential forms over Minkowski spacetime. The spacetime algebra is the typical fiber of the bundle (Rodrigues *et al.*, 1989; Rodrigues and Figueiredo, 1990; Maia *et al.*, 1990). Let $\{e_{\mu}\} \in \sec TM$ be a basis of TM and $\{\gamma^{\mu}\} \in \sec T^{*}M = \sec \Lambda^{1}M \subset \sec \mathscr{C}\ell(M, \hat{g})$ the dual basis, satisfying $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\nu\nu}$; $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The Dirac operator ∂ acting on sections of $\mathscr{C}\ell(M, \hat{g})$ is $\partial = d - \delta$, where d is the differential and δ the Hodge codifferential operator, and $\partial = d - \delta = \gamma^{\mu}\nabla_{\mu}$, where ∇ is the Levi-Civita connection of $g = \eta_{\mu\nu}\gamma^{\mu} \otimes \gamma^{\nu}$ ($\hat{g} = \eta^{\mu\nu} e_{\mu} \otimes e_{\nu}$). We can choose for simplicity $\{\gamma^{\mu}\}$ such that $\nabla_{\mu} = \partial_{\mu}$; thus $\partial = \gamma^{\mu}\partial_{\mu}$. Now, in this formalism, the *free* Maxwell equations $dF = 0$ and $\delta F = 0$ can be written as

$$\partial F = 0 \tag{10}$$

where the electromagnetic field $F \in \sec \Lambda^{2}(M) \subset \sec \mathscr{C}\ell(M, \hat{g})$. This form of the Maxwell equations is originally due to Riesz (1958). On the other hand, the Dirac equation for a free electron can be written in this formalism as

$$\partial\psi\gamma^{1}\gamma^{2} + \frac{mc}{\hbar}\psi\gamma^{0} = 0 \tag{11}$$

which is due to Hestenes (1966, 1967, 1975). The DH spinor field $\psi \in \sec(\Lambda^0(M)+\Lambda^2(M)+\Lambda^4(M)) \subset \sec \mathscr{C}\ell(M, \hat{g})$ can be written in the canonical form

$$\psi = \rho^{1/2}\, e^{\gamma^5\beta/2} R \tag{12}$$

where $\rho, \beta \in \sec \Lambda^0(M) \subset \sec \mathscr{C}\ell(M, \hat{g})$, and $\forall x \in M$, $R \in \mathrm{Spin}_+(1,3) \simeq SL(2, \mathbb{C})$, i.e., $RR^* = R^*R = 1$, where $*$ (called reversion) is the principal antiautomorphism in $\mathbb{R}_{1,3}$ (Figueiredo *et al.*, 1990*a,b*). Finally $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is the volume element.

Now, if we look for a solution of the free Maxwell equations (10) having the form given by equation (1), a simple substitution of equation (1) in equation (10) gives equation (8). Despite the fact that equation (2) is a nonlinear equation, we have shown (Vaz and Rodrigues, 1992) that it possesses plane wave solutions that satisfy the DH equation (11), namely:

$$\psi_- = \rho^{1/2}\, e^{\gamma^5\beta/2}\, e^{-\gamma^2\gamma^1(p\cdot x/\hbar)} \tag{13}$$

$$\psi_+ = \rho^{1/2}\, e^{\gamma^5\beta/2} \gamma^1\gamma^2\, e^{\gamma^2\gamma^1(p\cdot x/\hbar)} \tag{14}$$

which were identified (with $\beta = 0$ for ψ_+ and $\beta = \pi$ for ψ_-) as the "electron" and "positron" solutions, respectively. We have also proved that ψ_{e^-} and ψ_{e^+} rotate around the streamlines of an "electromagnetic fluid" with the same frequency $\omega_0 = 2mc^2/\hbar$ in different rotation senses and that such a rotation motion is the origin of mass in this theory—a conclusion also obtained by Hestenes (1991), but with a different point of view. In Rodrigues and Vaz (1992) we generalized our approach in order to obtain localized "electron" solutions by proving that each component $\phi_0\, e^{-i(p\cdot x/\hbar)}$ of the DH spinor field satisfies a nonlinear Klein–Gordon equation with nonlinearity of quantum potential type, that is,

$$\Box\psi + \left(\frac{mc}{\hbar}\right)^2 \psi = \frac{\Box\phi_0}{\phi_0}\, \psi \tag{15}$$

Our approach to these localized solutions is to be compared with the ones of Gueret and Vigier (1982*a–c*), Mackinnon (1978, 1981), and Barut (1990). Moreover, since the ψ field in our approach *is* of electromagnetic nature, we have an intimate relationship between ψ and the behavior of phase-locked cavities studied by Jennison (1978) and which have the inertial properties of classical particles.

3. PROOF OF EQUIVALENCE BETWEEN EQUATIONS (2) AND (4)

In order to prove that equations (2) and (4) are the same equation, we must prove that the components of the spinor field ψ satisfying equation

(2) and the components of the spinor field Ψ satisfying equation (4) satisfy the same equation. Thus we shall first write equation (4) in terms of its components, then we shall write equation (2) in terms of its components, and finally we compare them.

Let us calculate the term in braces in equation (4). For

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, -\bar{\psi}_3, -\bar{\psi}_4) \tag{16}$$

we have that (the bar over the components denotes complex conjugation)

$$\begin{aligned} \operatorname{Im}(\partial_\mu \bar{\Psi}\Psi) = \frac{-i}{2} [& (\partial_\mu \bar{\psi}_1 \psi_1 - \bar{\psi}_1 \partial_\mu \psi_1) + (\partial_\mu \bar{\psi}_2 \psi_2 - \bar{\psi}_2 \partial_\mu \psi_2) \\ & - (\partial_\mu \bar{\psi}_3 \psi_3 - \bar{\psi}_3 \partial_\mu \psi_3) - (\partial_\mu \bar{\psi}_4 \psi_4 - \bar{\psi}_4 \partial_\mu \psi_4)] \equiv \frac{-i}{2} \xi \end{aligned} \tag{17}$$

$$\begin{aligned} \operatorname{Im}(\partial_\mu \bar{\Psi}\gamma^5\Psi) = \frac{-1}{2} [& (\partial_\mu \bar{\psi}_1 \psi_3 + \bar{\psi}_3 \partial_\mu \psi_1) + (\partial_\mu \bar{\psi}_2 \psi_4 + \bar{\psi}_4 \partial_\mu \psi_2) \\ & - (\partial_\mu \bar{\psi}_3 \psi_1 - \bar{\psi}_1 \partial_\mu \psi_3) - (\partial_\mu \bar{\psi}_4 \psi_2 + \bar{\psi}_2 \partial_\mu \psi_4)] \equiv \frac{-1}{2} \eta \end{aligned} \tag{18}$$

and then

$$\{\operatorname{Im}(\partial_\mu \bar{\Psi}\Psi) - \gamma^5 \operatorname{Im}(\partial_\mu \bar{\Psi}\gamma^5\Psi)\}\Psi = \frac{-i}{2} \left(\xi \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} + \eta \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \right) \tag{19}$$

Thus equation (4) is explicitly

$$\gamma^\mu \partial_\mu \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = -\frac{1}{2} \gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \begin{pmatrix} \xi\psi_1 + \eta\psi_3 \\ \xi\psi_2 + \eta\psi_4 \\ \xi\psi_3 + \eta\psi_1 \\ \xi\psi_4 + \eta\psi_2 \end{pmatrix} = -\frac{1}{2} \gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} \tag{20}$$

Now we shall write equation (2) in terms of components. In the Appendix we show that a DH spinor field has the following matrix representation:

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix}, \qquad \psi^* = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\psi_2 & \psi_1 & -\psi_4 & \psi_3 \\ -\bar{\psi}_3 & -\bar{\psi}_4 & \bar{\psi}_1 & \bar{\psi}_2 \\ -\psi_4 & \psi_3 & -\psi_2 & \psi_1 \end{pmatrix} \tag{21}$$

Since we also have

$$\gamma^1\gamma^2 = -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{22}$$

the calculation of $\mathscr{F}(\psi)$ is a simple task. We have

$$\psi\gamma^1\gamma^2 = -i\begin{pmatrix} \psi_1 & \bar{\psi}_2 & \psi_3 & -\bar{\psi}_4 \\ \psi_2 & -\bar{\psi}_1 & \psi_4 & \bar{\psi}_3 \\ \psi_3 & -\bar{\psi}_4 & \psi_1 & \bar{\psi}_2 \\ \psi_4 & \bar{\psi}_3 & \psi_2 & -\bar{\psi}_1 \end{pmatrix} \tag{23}$$

Also,

$$\begin{aligned} \psi\gamma^1\gamma^2\partial_\mu\psi^* &= -i\begin{pmatrix} \psi_1 & \bar{\psi}_2 & \psi_3 & -\bar{\psi}_4 \\ \psi_2 & -\bar{\psi}_1 & \psi_4 & \bar{\psi}_3 \\ \psi_3 & -\bar{\psi}_4 & \psi_1 & \bar{\psi}_2 \\ \psi_4 & \bar{\psi}_3 & \psi_2 & -\bar{\psi}_1 \end{pmatrix}\begin{pmatrix} \partial_\mu\bar{\psi}_1 & \partial_\mu\bar{\psi}_2 & -\partial_\mu\bar{\psi}_3 & -\partial_\mu\bar{\psi}_4 \\ -\partial_\mu\psi_2 & \partial_\mu\psi_1 & -\partial_\mu\psi_4 & \partial_\mu\psi_3 \\ -\partial_\mu\bar{\psi}_3 & -\partial_\mu\bar{\psi}_4 & \partial_\mu\bar{\psi}_1 & \partial_\mu\bar{\psi}_2 \\ -\partial_\mu\psi_4 & \partial_\mu\psi_3 & -\partial_\mu\psi_2 & \partial_\mu\psi_1 \end{pmatrix} \\ &= -i\begin{pmatrix} A & B & C & D \\ \bar{B} & -\bar{A} & -\bar{D} & \bar{C} \\ C & D & A & B \\ -\bar{D} & \bar{C} & \bar{B} & -\bar{A} \end{pmatrix} \end{aligned} \tag{24}$$

where we have defined

$$A = \psi_1\partial_\mu\bar{\psi}_1 - \bar{\psi}_2\partial_\mu\psi_2 - \psi_3\partial_\mu\bar{\psi}_3 + \bar{\psi}_4\partial_\mu\psi_4 \tag{25}$$

$$B = \psi_1\partial_\mu\bar{\psi}_2 + \bar{\psi}_2\partial_\mu\psi_1 - \psi_3\partial_\mu\bar{\psi}_4 - \bar{\psi}_4\partial_\mu\psi_3 \tag{26}$$

$$C = -\psi_1\partial_\mu\bar{\psi}_3 - \bar{\psi}_2\partial_\mu\psi_4 + \psi_3\partial_\mu\bar{\psi}_1 + \bar{\psi}_4\partial_\mu\psi_2 \tag{27}$$

$$D = -\psi_1\partial_\mu\bar{\psi}_4 + \bar{\psi}_2\partial_\mu\psi_3 + \psi_3\partial_\mu\bar{\psi}_2 - \bar{\psi}_4\partial_\mu\psi_1 \tag{28}$$

Since $\psi\gamma^2\gamma^1 = -\psi\gamma^1\gamma^2$ we have, using equations (24) and (23),

$$\begin{aligned} \psi\gamma^1\gamma^2(\partial_\mu\psi^*)\psi\gamma^2\gamma^1 &= \begin{pmatrix} A & B & C & D \\ \bar{B} & -\bar{A} & -\bar{D} & \bar{C} \\ C & D & A & B \\ -\bar{D} & \bar{C} & \bar{B} & -\bar{A} \end{pmatrix}\begin{pmatrix} \psi_1 & \bar{\psi}_2 & \psi_3 & -\bar{\psi}_4 \\ \psi_2 & -\bar{\psi}_1 & \psi_4 & \bar{\psi}_3 \\ \psi_3 & -\bar{\psi}_4 & \psi_1 & \bar{\psi}_2 \\ \psi_4 & +\bar{\psi}_3 & \psi_2 & -\bar{\psi}_1 \end{pmatrix} \\ &= \begin{pmatrix} E & F & G & H \\ -\bar{F} & \bar{E} & \bar{H} & -\bar{G} \\ G & H & E & F \\ \bar{H} & -\bar{G} & -\bar{F} & \bar{E} \end{pmatrix} = \Phi \end{aligned} \tag{29}$$

where we used

$$E = A\psi_1 + B\psi_2 + C\psi_3 + D\psi_4 \tag{30}$$

$$F = A\bar{\psi}_2 - B\bar{\psi}_1 - C\bar{\psi}_4 + D\bar{\psi}_3 \tag{31}$$

$$G = A\psi_3 + B\psi_4 + C\psi_1 + D\psi_2 \tag{32}$$

$$H = -A\bar{\psi}_2 + B\bar{\psi}_3 + C\bar{\psi}_2 - D\bar{\psi}_1 \tag{33}$$

Finally, since $\psi\psi^* = \rho e^{\gamma^5\beta}$ and using equation (29) we have for equation (2)

$$\gamma^\mu \partial_\mu \psi = -\gamma^\mu \frac{e^{-\gamma^5\beta}}{\rho} \Phi \tag{34}$$

Now if we introduce the expressions for A, B, C, D into the ones for E, F, G, H and remember that

$$\Omega_1 = \psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 - \psi_3\bar{\psi}_3 - \psi_4\bar{\psi}_4 \tag{35}$$

$$\Omega_2 = i(\psi_1\bar{\psi}_3 + \psi_2\bar{\psi}_4 - \psi_3\bar{\psi}_1 - \psi_4\bar{\psi}_2) \tag{36}$$

then after a straightforward but tedious calculation we arrive at

$$\phi_1 = E = \xi\psi_1 + \eta\psi_3 + \Omega_1\partial_\mu\psi_1 + i\Omega_2\partial_\mu\psi_3 \tag{37}$$

$$\phi_2 = -\bar{F} = \xi\psi_2 + \eta\psi_4 + \Omega_1\partial_\mu\psi_2 + i\Omega_2\partial_\mu\psi_4 \tag{38}$$

$$\phi_3 = G = \xi\psi_3 + \eta\psi_1 + \Omega_1\partial_\mu\psi_3 + i\Omega_2\partial_\mu\psi_1 \tag{39}$$

$$\phi_4 = \bar{H} = \xi\psi_4 + \eta\psi_2 + \Omega_1\partial_\mu\psi_4 + i\Omega_2\partial_\mu\psi_2 \tag{40}$$

and

$$\begin{aligned}
\Phi &= \begin{pmatrix} \phi_1 & -\bar{\phi}_2 & \phi_3 & \bar{\phi}_4 \\ \phi_2 & \bar{\phi}_1 & \phi_4 & -\bar{\phi}_3 \\ \phi_3 & \bar{\phi}_4 & \phi_1 & -\bar{\phi}_2 \\ \phi_4 & -\bar{\phi}_2 & \phi_2 & \bar{\phi}_1 \end{pmatrix} \\
&= \begin{pmatrix} \chi_1 & -\bar{\chi}_2 & \chi_3 & \bar{\chi}_4 \\ \chi_2 & \bar{\chi}_1 & \chi_4 & -\bar{\chi}_3 \\ \chi_3 & \bar{\chi}_4 & \chi_1 & -\bar{\chi}_2 \\ \chi_4 & -\bar{\chi}_3 & \chi_2 & \bar{\chi}_1 \end{pmatrix} + \Omega_1\partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} \\
&\quad + i\Omega_2\partial_\mu \begin{pmatrix} \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \\ \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \end{pmatrix}
\end{aligned} \tag{41}$$

The last matrix can be written as

$$\begin{pmatrix} \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \\ \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \end{pmatrix} = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} \tag{42}$$

in such a way that the sum of the two latter matrices in equation (41) is

$$\begin{aligned} &\Omega_1 \partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} \\ &\quad + \Omega_2 \begin{pmatrix} & & i & 0 \\ & & 0 & i \\ i & 0 & & \\ 0 & i & & \end{pmatrix} \partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} \\ &= (\Omega_1 - \gamma^5 \Omega_2) \partial_\mu \psi \end{aligned} \tag{43}$$

Now if we use equation (21) we find that

$$\psi\psi^* = \begin{pmatrix} \Omega_1 & 0 & i\Omega_2 & 0 \\ 0 & \Omega_1 & 0 & i\Omega_2 \\ i\Omega_2 & 0 & \Omega_1 & 0 \\ 0 & i\Omega_2 & 0 & \Omega_1 \end{pmatrix} = \Omega_1 - \gamma^5 \Omega_2 \tag{44}$$

and using equation (44) in equation (43) and then in equation (41) we have that

$$\Phi = \chi + (\psi\psi^*)\partial_\mu \psi = \chi + \rho e^{\gamma^5 \beta} \partial_\mu \psi \tag{45}$$

where χ is the first matrix on the rhs of equation (41).

Introducing this result in equation (32), we get

$$\gamma^\mu \partial_\mu \psi = -\frac{1}{2} \gamma^\mu \frac{e^{-\gamma^5 \beta}}{\rho} \chi \tag{46}$$

or, in terms of components,

$$\gamma^\mu \partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} = -\frac{1}{2} \gamma^\mu \frac{e^{-\gamma^5 \beta}}{\rho} \begin{pmatrix} \chi_1 & -\bar{\chi}_2 & \chi_3 & \bar{\chi}_4 \\ \chi_2 & \bar{\chi}_1 & \chi_4 & -\bar{\chi}_3 \\ \chi_3 & \bar{\chi}_4 & \chi_1 & -\bar{\chi}_2 \\ \chi_4 & -\bar{\chi}_3 & \chi_2 & \bar{\chi}_1 \end{pmatrix} \tag{47}$$

Now we compare equations (20) and (47): they are the same equation![5] Thus we have proved that equations (2) and (4) are one and the same equation. The reader is invited to compare our approach to equation (2) in Vaz and Rodrigues (1992)—just one line of calculus— with that one of Campolattaro (1990) to equation (4)—six pages of calculus.

Note the identification of the "complexion" α with β, whose values $\beta = 0$ and $\beta = \pi$ distinguish electrons from positrons in the Dirac theory (Vaz and Rodrigues, 1992; Hestenes, 1991).

4. PLANE WAVE SOLUTIONS OF THE NONLINEAR EQUATION

It is only a matter of verification to see that plane waves are solutions of equation (20) or equation (47). To simplify the calculations, we shall consider the rest frame. The equations for the components are

$$\partial_0 \psi_1 = -\frac{1}{2\rho}\left[\cos\beta(\xi\psi_1 + \eta\psi_3) - i\sin\beta(\xi\psi_3 + \eta\psi_1)\right] \tag{48}$$

$$\partial_0 \psi_2 = -\frac{1}{2\rho}\left[\cos\beta(\xi\psi_2 + \eta\psi_4) - i\sin\beta(\xi\psi_4 + \eta\psi_2)\right] \tag{49}$$

$$\partial_0 \psi_3 = -\frac{1}{2\rho}\left[\cos\beta(\xi\psi_3 + \eta\psi_1) - i\sin\beta(\xi\psi_1 + \eta\psi_3)\right] \tag{50}$$

$$\partial_0 \psi_4 = -\frac{1}{2\rho}\left[\cos\beta(\xi\psi_4 + \eta\psi_2) - i\sin\beta(\xi\psi_2 + \eta\psi_4)\right] \tag{51}$$

where, in accordance with equations (17) and (18),

$$\begin{aligned}\xi = {} & (\partial_\mu\bar{\psi}_1\psi_1 - \bar{\psi}_1\partial_\mu\psi_1) + (\partial_\mu\bar{\psi}_2\psi_2 - \bar{\psi}_2\partial_\mu\psi_2) \\ & - (\partial_\mu\bar{\psi}_3\psi_3 - \bar{\psi}_3\partial_\mu\psi_3) - (\partial_\mu\bar{\psi}_4\psi_4 - \bar{\psi}_4\partial_\mu\psi_4)\end{aligned} \tag{52}$$

$$\begin{aligned}\eta = {} & (\partial_\mu\bar{\psi}_1\psi_3 - \bar{\psi}_3\partial_\mu\psi_1) + (\partial_\mu\bar{\psi}_2\psi_4 - \bar{\psi}_4\partial_\mu\psi_2) \\ & - (\partial_\mu\bar{\psi}_3\psi_1 + \bar{\psi}_1\partial_\mu\psi_3) - (\partial_\mu\bar{\psi}_4\psi_2 + \bar{\psi}_2\partial_\mu\psi_4)\end{aligned} \tag{53}$$

Now it is trivial to verify that

$$\Psi_1^{(-)} = \sqrt{\rho}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imc^2t/\hbar}; \qquad \Psi_2^{(-)} = \sqrt{\rho}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-imc^2t/\hbar} \tag{54}$$

[5] More precisely, each column of equation (17) contains the same information as equation (20).

are solutions of the above equations (48)-(51) provided that

$$\beta = 0 \tag{55}$$

and that

$$\Psi_1^{(+)} = \sqrt{\rho}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{imc^2t/\hbar}; \qquad \Psi_2^{(+)} = \sqrt{\rho}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{imc^2t/\hbar} \tag{56}$$

are also solutions of equations (48)-(51) provided that

$$\beta = \pi \tag{57}$$

These solutions (54)-(56) are written in the spacetime algebra as

$$\psi_1^{(-)} = \sqrt{\rho}\; e^{\gamma^5\beta/2}\; e^{-\gamma^2\gamma^1 mc^2t/\hbar}\big|_{\beta=0} \tag{58}$$

$$\psi_2^{(-)} = \sqrt{\rho}\; e^{\gamma^5\beta/2}\gamma^3\gamma^1\; e^{-\gamma^2\gamma^1 mc^2t/\hbar}\big|_{\beta=0} \tag{59}$$

$$\psi_1^{(+)} = \sqrt{\rho}\, e^{\gamma^5\beta/2}\gamma^1\gamma^2\; e^{\gamma^2\gamma^1 mc^2t/\hbar}\big|_{\beta=\pi} \tag{60}$$

$$\psi_2^{(+)} = \sqrt{\rho}\; e^{\gamma^5\beta/2}\gamma^3\gamma^1\gamma^1\gamma^2\; e^{\gamma^2\gamma^1 mc^2t/\hbar}\big|_{\beta=\pi} \tag{61}$$

which can be easily verified by right multiplication by the idempotent $e = \frac{1}{2}(1+\gamma^0)$ according to the method described in Rodrigues and Oliveira (1990).

5. CONCLUSIONS

We cannot but be surprised by the fact that equation (4) exhibits plane solutions. Indeed it is very difficult to deal with it, so that the proof requires an extensive calculus. On the other hand, the same equation in the Clifford bundle, equation (2), is simple, and the fact that it exhibits plane wave solutions can be easily seen in this case. The essential difference between our method and the one of Campolattaro is that a DH spinor field can be represented as a matrix which has an inverse, while the SDC spinor field used by Campolattaro is represented by a one-column matrix which does not have an inverse. Thus Campolattaro's method is expected to be more complicated.

APPENDIX A. MATRIX REPRESENTATION OF THE DIRAC–HESTENES SPINOR FIELD

In this Appendix we give a matrix representation for the DH spinor field. In the Dirac representation we have

$$\gamma^0=\begin{pmatrix}1&0&&\\0&1&&\\&&-1&0\\&&0&-1\end{pmatrix}=\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix}$$

$$\gamma^1=\begin{pmatrix}&&0&1\\&&1&0\\0&-1&&\\-1&0&&\end{pmatrix}=\begin{pmatrix}0&\sigma_1\\-\sigma_1&0\end{pmatrix}$$

$$\gamma^2=\begin{pmatrix}&&0&-i\\&&i&0\\0&i&&\\-i&0&&\end{pmatrix}=\begin{pmatrix}0&\sigma_2\\-\sigma_2&0\end{pmatrix}$$

$$\gamma^3=\begin{pmatrix}&&1&0\\&&0&-1\\-1&0&&\\0&1&&\end{pmatrix}=\begin{pmatrix}0&\sigma_3\\-\sigma_3&0\end{pmatrix}$$

Since $\psi\in\sec(\Lambda^0(M)+\Lambda^2(M)+\Lambda^4)\subset\sec\mathcal{C}\ell(M,\hat{g})$, we have that

$$\psi=a1+a_{01}\gamma^0\gamma^1+a_{02}\gamma^0\gamma^2+a_{03}\gamma^0\gamma^3+a_{12}\gamma^1\gamma^2$$
$$+a_{13}\gamma^1\gamma^3+a_{23}\gamma^2\gamma^3+a_{0123}\gamma^0\gamma^1\gamma^2\gamma^3$$

but in this representation

$$\gamma^0\gamma^1=\begin{pmatrix}0&\sigma_1\\\sigma_1&0\end{pmatrix};\qquad \gamma^0\gamma^2=\begin{pmatrix}0&\sigma_2\\\sigma_2&0\end{pmatrix};\qquad \gamma^0\gamma^3=\begin{pmatrix}0&\sigma_3\\\sigma_3&0\end{pmatrix}$$

$$\gamma^1\gamma^2=-i\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix};\qquad \gamma^1\gamma^3=i\begin{pmatrix}\sigma_2&0\\0&\sigma_2\end{pmatrix};\qquad \gamma^2\gamma^3=-i\begin{pmatrix}\sigma_1&0\\0&\sigma_1\end{pmatrix}$$

$$\gamma^0\gamma^1\gamma^2\gamma^3=\gamma^5=-i\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Thus we have

$$\psi=\begin{pmatrix}\psi_1&-\bar{\psi}_2&\psi_3&\bar{\psi}_4\\\psi_2&\bar{\psi}_1&\psi_4&-\bar{\psi}_3\\\psi_3&\bar{\psi}_4&\psi_1&-\bar{\psi}_2\\\psi_4&-\bar{\psi}_3&\psi_2&\bar{\psi}_1\end{pmatrix}$$

where

$$\psi_1 = a_0 - ia_{12}$$

$$\psi_2 = -a_{13} - ia_{23}$$

$$\psi_3 = a_{03} - ia_{0123}$$

$$\psi_4 = a_{01} + ia_{02}$$

For $\psi^* = \gamma^0\psi^+\gamma^0$ we have then

$$\psi^* = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\psi_2 & \psi_1 & -\bar{\psi}_4 & \psi_3 \\ -\bar{\psi}_3 & -\bar{\psi}_4 & \bar{\psi}_1 & \bar{\psi}_2 \\ -\psi_4 & \psi_3 & -\bar{\psi}_2 & \psi_1 \end{pmatrix}$$

APPENDIX B. THEOREM OF RAINICH–MISNER–WHEELER

In this Appendix we give a proof in terms of spacetime algebra of the theorem of Rainich-Misner-Wheeler (Rainich, 1927; Misner and Wheeler, 1957) because of its fundamental importance for this work.

If we define an extremal field as a field for which the magnetic (electric) field is zero and the electric (magnetic) field is parallel to one coordinate axis, the theorem of Rainich-Misner-Wheeler says that: "At any point of Minkowski spacetime any nonnull electromagnetic field can be reduced to an extremal field by a Lorentz transformation and a duality rotation."

Let $F = \frac{1}{2}F_{\mu\nu}\gamma^\mu \wedge \gamma^\nu \in \sec \Lambda^2(M) \subset \sec \mathscr{Cl}(M, \hat{g})$ be the electromagnetic field. The invariants of F are given by $F^2 = F \cdot F + F \wedge F$. In terms of Pauli algebra $\mathbb{R}_{3,0} \simeq \mathbb{R}^+_{1,3}$ we have $F = \mathbf{E} + \hat{i}\mathbf{H}$, where $\mathbf{E} = E_i\boldsymbol{\sigma}_i$, $\mathbf{H} = H_i\boldsymbol{\sigma}_i$, $\boldsymbol{\sigma}^i = \gamma^i\gamma^0$, $\hat{i} = \boldsymbol{\sigma}^1\boldsymbol{\sigma}^2\boldsymbol{\sigma}^3$ (the $\boldsymbol{\sigma}^i$ are the generators of $\mathbb{R}_{3,0}$), and we have

$$F \cdot F = \mathbf{E}^2 - \mathbf{H}^2; \qquad F \wedge F = 2\hat{i}\mathbf{E} \cdot \mathbf{H}$$

Let us consider a duality rotation of F by an angle α, that is,

$$F' = e^{\gamma^5\alpha}F = \cos\alpha F + \gamma^5 F \sin\alpha$$

The invariants of the fields are changed under a duality rotation in such a way that from

$$F'^2 = e^{2\gamma^5\alpha}F^2$$

we have

$$F' \cdot F' = \cos 2\alpha(F \cdot F) + \sin 2\alpha\gamma^5(F \wedge F)$$

$$F' \wedge F' = \cos 2\alpha(F \wedge F) + \sin 2\sigma\gamma^5(F \cdot F)$$

or, by writing $F=\mathbf{E}'+\hat{i}\mathbf{H}'$,

$$\mathbf{E}'^2-\mathbf{H}'^2=\cos 2\alpha(\mathbf{E}^2-\mathbf{H}^2)-\sin 2\alpha 2(\mathbf{E}\cdot\mathbf{H})$$
$$2(\mathbf{E}'\cdot\mathbf{H}')=\cos 2\alpha 2(\mathbf{E}\cdot\mathbf{H})-\sin 2\alpha(\mathbf{E}^2-\mathbf{H}^2)$$

Now we can choose α in such a way that

$$\mathbf{E}'\cdot\mathbf{H}'=0$$

that is,

$$\tan 2\alpha=\frac{\gamma^5(F\wedge F)}{(F\cdot F)}=\frac{2(\mathbf{E}\cdot\mathbf{H})}{\mathbf{H}^2-\mathbf{E}^2}$$

and then for $F'\cdot F'$ we have

$$\mathbf{E}'^2-\mathbf{H}'^2=\pm[(\mathbf{E}^2-\mathbf{H}^2)^2+4(\mathbf{E}\cdot\mathbf{H})^2]^{1/2}$$

where the different signs come from the fact that $\tan\phi$ has period π and $\cos(\phi+\pi)=-\cos\phi$, so that the angles ϕ and $\phi+\pi$ correspond respectively to $\mathbf{E}'^2-\mathbf{H}'^2<0$ and $\mathbf{E}'^2-\mathbf{H}'^2>0$. Indeed, since $2\alpha=\phi+\pi$ and $\alpha=\phi/2+\pi/2$, so that

$$e^{\gamma^5\alpha}=e^{\gamma^5\phi/2}\,e^{\gamma^5\pi/2}=e^{\gamma^5\phi/2}\gamma^5$$

the duality rotation by $\pi/2$, i.e., $e^{\gamma^5\pi/2}=\gamma^5$, transforms an electric field into a magnetic one and vice versa.

Now, a well-known (Doubrovine *et al.*, 1987) theorem says that: "If $\mathbf{E}'\cdot\mathbf{H}'=0$, then there exists a Lorentz transformation R such that $F''=RF'R^*=\mathbf{E}''+\hat{i}\mathbf{H}''$ and we have (a) if $\mathbf{E}'^2-\mathbf{H}'^2>0$, then $\mathbf{E}''\neq 0$ and $\mathbf{H}''=0$; (b) $\mathbf{E}'^2-\mathbf{H}'^2<0$, then $\mathbf{E}''=0$ and $\mathbf{H}''\neq 0$." Let us find this transformation explicitly. Consider $\mathbf{v}=v\boldsymbol{\sigma}^1$; then for a Lorentz transformation $[\gamma=(1-\beta^2)^{-1/2}]$

$$E_1''=E_1';\qquad E_2''=\gamma(E_2'+\beta H_3');\qquad E_3''=\gamma(E_3'-\beta H_2')$$
$$H_1''=H_1';\qquad H_2''=\gamma(H_2'+\beta E_3');\qquad H_3''=\gamma(E_3'-\beta E_2')$$

Considering a duality rotation such that $\mathbf{E}'\cdot\mathbf{H}'=0$ and $\mathbf{E}'^2-\mathbf{H}'^2<0$ and choosing $\boldsymbol{\sigma}^2$ and $\boldsymbol{\sigma}^3$ such that $\mathbf{E}=E\boldsymbol{\sigma}^2$ and $\mathbf{H}=H\boldsymbol{\sigma}^3$ and $\beta=v/c=E/H$, we have

$$E_1''=E_2''=E_3''=0;\qquad H_1''=H_2''=0;\qquad H_3''=(H'^2-E'^2)^{1/2}$$

Thus, by defining

$$\bar{\alpha}=-\alpha,\qquad \bar{R}=R^*$$

we have shown that

$$[\exp(\gamma^5\bar{\alpha})]\bar{R}^*F\bar{R}=-\hat{i}H\boldsymbol{\sigma}^3=H\gamma^1\gamma^2$$

from which it follows that

$$F = [\exp(\gamma^5 \bar{\alpha})] \bar{R} (H \gamma^1 \gamma^2) \bar{R}^*$$

or, by writing

$$H = b\rho, \qquad \psi = \sqrt{\rho}\,[\exp(\gamma^5 \bar{\alpha}/2)] \bar{R}$$

that

$$F = b \psi \gamma^1 \gamma^2 \psi$$

ACKNOWLEDGMENTS

We are grateful to Prof. E. Recami and Drs. J. R. R. Zeni and Q. A. G. de Souza for discussions.

REFERENCES

Barut, A. O. (1990). *Physics Letters A*, **143**, 349.

Campolattaro, A. A. (1980*a*). *International Journal of Theoretical Physics*, **19**, 99.

Campolattaro, A. A. (1980*b*). *International Journal of Theoretical Physics*, **19**, 127.

Campolattaro, A. A. (1990). *International Journal of Theoretical Physics*, **29**, 141.

Doubrovine, B., Novikov, S., and Fomenko, A. (1987). *Géométrie Contemporaine, Méthodes et Applications*, Part 1, Mir, Moscow.

Figueiredo, V. L., de Oliveira, E. C., and Rodrigues, W. A., Jr. (1990*a*). *International Journal of Theoretical Physics*, **29**, 371.

Figueiredo, V. L., de Oliveira, E. C., and Rodrigues, W. A., Jr. (1990*b*). *Algebras Groups, and Geometries*, **7**, 153.

Gueret, P., and Vigier, J. P. (1982*a*). *Lettere Nuovo Cimento*, **35**, 256.

Gueret, P., and Vigier, J. P. (1982*b*). *Lettere Nuovo Cimento*, **35**, 260.

Gueret, P., and Vigier, J. P. (1982*c*). *Lettere Nuovo Cimento*, **38**, 125.

Hestenes, D. (1966). *Space-Time Algebra*, Gordon and Breach, New York.

Hestenes, D. (1967). *Journal of Mathematical Physics*, **8**, 789.

Hestenes, D. (1975). *Journal of Mathematical Physics*, **16**, 556.

Hestenes, D. (1991). *Foundations of Physics*, **20**, 1213.

Jennison, R. C. (1978). *Journal of Physics A*, **11**, 1525.

Mackinnon, L. (1978). *Foundations of Physics*, **8**, 157.

Mackinnon, L. (1981). *Lettere Nuovo Cimento*, **32**, 311.

Maia, A., Jr., Recami, E., Rosa, M. A. F., and Rodrigues, W. A., Jr. (1990). *Journal of Mathematical Physics*, **31**, 502.

Misner, C. W., and Wheeler, J. A. (1957). *Annals of Physics*, **2**, 525.

Rainich, G. Y. (1925). *Transactions of the American Mathematical Society*, **27**, 106.

Riesz, M. (1958). Clifford Number and Spinors, Lecture Series No. 38, Institute for Fluid Mechanics and Applied Mathematics, University of Maryland.

Rodrigues, W. A., Jr., and de Oliveira, E. C. (1990). *International Journal of Theoretical Physics*, **29**, 397.

Rodrigues, W. A., Jr., Faria-Rosa, M. A., Maia, A., Jr., and Recami, E. (1989). *Hadronic Journal*, **12**, 187.

Rodrigues, W. A., Jr., and Figueiredo, V. L. (1990). *International Journal of Theoretical Physics*, **29**, 413.

Rodrigues, W. A., Jr., and Vaz, J., Jr. (1992). Nondispersive de Broglie wavepackets from the free Maxwell equations, submitted.

Vaz, J., Jr., and Rodrigues, W. A., Jr. (1992). Particle and antiparticle configurations of the free Maxwell equations satisfying the Dirac equation, submitted.